Int J Theor Phys (2011) 50: 233–243
DOI 10.1007/s10773-010-0514-z

# Fedosov Quantization of Fractional Lagrange Spaces

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Received: 24 July 2010 / Accepted: 23 September 2010 / Published online: 8 October 2010


**Abstract** The main goal of this work is to perform a nonholonomic deformation (Fedosov type) quantization of fractional Lagrange–Finsler geometries. The constructions are provided for a fractional almost Kähler model encoding equivalently all data for fractional Euler–Lagrange equations with Caputo fractional derivative.

**Keywords** Fractional Finsler geometry · Lagrange space · Almost Kähler space · Deformation quantization · Fractional Fedosov space

## 1 Introduction

This paper is a "quantum" partner work (in the meaning of deformation quantization) of the article [1] on almost Kähler models of fractional Lagrange–Finsler geometries. It belongs to a series of our works on fractional (i.e. non-integer dimension) nonholonomic spaces, theirs Ricci flows and certain fractional type gravity and geometric mechanics models [2, 3].[1]

[1] Readers are recommended to consult the main results and conventions in advance.

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There are some preliminary attempts (see, for instance, [4–6]) to quantize fractional mechanical models and field interactions as generalized quantum mechanics and related fractional quantum field theories. Such constructions are for some particular cases of fractional calculus and physical models [7–15]. It is not clear if and how a general formalism encoding quantum fractional theories can be elaborated.

Following the geometry of nonholonomic distributions modelling geometries of noninteger (i.e. fractional) dimensions, a self-consistent quantization formalism can be elaborated following the Fedosov deformation quantization [16, 17]. The original constructions were provided for classical and quantum Kähler geometries. Latter, the approach was generalized for almost Kähler geometries [18] which allowed to include into the quantization deformation scheme various types of Lagrange–Finsler, Hamilton–Cartan and Einstein spaces and generalizations, see papers [19–22] and references therein.

A geometrization of fractional calculus and various types of fractional mechanical and field theories is possible for the so-called Caputo fractional derivative, see details in [2, 3]. For fractional regular Lagrange mechanical models, such effective geometries can be derived as almost Kähler configurations with more "rich" geometric nonholonomic and/or fractional structures [1]. This suggests a "realistic" possibility to quantize in general form, following methods of deformation quantization, various types of fractional geometries and physical theories which via nonholonomic deformations can be re-defined as some types of almost Kähler spaces. In this article, we show how to perform such a program for fractional Lagrange spaces.

This work is organized in the form: In Sect. 2, we remember the most important properties and formulas on Caputo fractional derivatives and related nonholonomic (co) frame formalism for fractional tangent bundles. Section 3 is devoted to definition of Fedosov operators for fractional Lagrange spaces. The main results on deformation quantization of fractional Lagrange mechanics are provided in Sect. 4. Finally, we conclude and discuss the results in Sect. 5.

## 2 Preliminaries: Fractional Calculus and Almost Kähler Geometry

We outline some necessary formulas and results on fractional calculus and nonholonomic geometry, see details and notation conventions in [1–3].[2]

### 2.1 Caputo Fractional Derivatives

There is a class of fractional derivatives which resulting in zero for actions on constants. This property is crucial for constructing geometric models of theories with fractional calculus.

We define and denote the fractional left, respectively, right Caputo derivatives in the forms

[2] We use "up" and "low" left labels which are convenient to be introduced in order to not create confusions with a number of "horizontal" and "vertical" right indices and labels which must be distinguished if the manifolds are provided with N-connection structure. In our papers, we work with mixed sets of "fractional" and "integer" dimensions (and holonomic and nonholonomic variables etc.). This makes the systems of labels and notations for geometric objects to be quite sophisticate even in coordinate free form formalisms. Unfortunately, further simplifications seem to be not possible.

$$
\begin{aligned}
{}_{1x}\overset{\alpha}{\underline{\partial}}_x f(x) &:= \frac{1}{\Gamma(s-\alpha)} \int_{1x}^{x} (x-x')^{s-\alpha-1} \left(\frac{\partial}{\partial x'}\right)^s f(x')dx'; \\
{}_{x}\overset{\alpha}{\underline{\partial}}_{2x} f(x) &:= \frac{1}{\Gamma(s-\alpha)} \int_{x}^{2x} (x'-x)^{s-\alpha-1} \left(-\frac{\partial}{\partial x'}\right)^s f(x')dx'.
\end{aligned}
\tag{1}
$$

The fractional absolute differential $\overset{\alpha}{d}$, corresponding to above fractional derivatives, is written $\overset{\alpha}{d} := (dx^j)^\alpha {}_0\overset{\alpha}{\underline{\partial}}_j$, where $\overset{\alpha}{d}x^j = (dx^j)^\alpha \frac{(x^j)^{1-\alpha}}{\Gamma(2-\alpha)}$, where we consider ${}_1x^i = 0$.

For a fractional tangent bundle $\overset{\alpha}{\underline{T}}M$ for $\alpha \in (0,1)$, associated to a manifold $M$ of necessary smooth class and integer $\dim M = n$, we write both the integer and fractional local coordinates in the form $u^\beta = (x^j, y^a)$. The symbol $T$ is underlined in order to emphasize that we shall associate the approach to a fractional Caputo derivative. A fractional frame basis $\overset{\alpha}{\underline{e}}_\beta = e^{\beta'}_\beta(u^\beta)\overset{\alpha}{\underline{\partial}}_{\beta'}$ on $\overset{\alpha}{\underline{T}}M$ is connected via a Vielbein transform $e^{\beta'}_\beta(u^\beta)$ with a fractional local coordinate basis

$$
\overset{\alpha}{\underline{\partial}}_{\beta'} = \left( \overset{\alpha}{\underline{\partial}}_{j'} = {}_{1x^{j'}}\overset{\alpha}{\underline{\partial}}_{j'}, \overset{\alpha}{\underline{\partial}}_{b'} = {}_{1y^{b'}}\overset{\alpha}{\underline{\partial}}_{b'} \right),
\tag{2}
$$

for $j' = 1, 2, \ldots, n$ and $b' = n+1, n+2, \ldots, n+n$. The fractional co-bases are written $\overset{\alpha}{\underline{e}}{}^\beta = e^\beta_{\beta'}(u^\beta)\overset{\alpha}{d}u^{\beta'}$, where the fractional local coordinate co-basis is

$$
\overset{\alpha}{d}u^{\beta'} = ((dx^{i'})^\alpha, (dy^{a'})^\alpha).
\tag{3}
$$

Explicit constructions in the geometry of fractional tangent bundle depend on the type of chosen fractional derivative.

### 2.2 A Geometrization of Fractional Lagrange Mechanics

A fractional Lagrange space $\overset{\alpha}{\underline{L^n}} = (\overset{\alpha}{\underline{M}}, \overset{\alpha}{L})$ of fractional dimension $\alpha \in (0,1)$, for a regular real function $\overset{\alpha}{L} : \overset{\alpha}{\underline{T}}M \to \mathbb{R}$, is associated to a prime Lagrange space $L^n = (M, L)$, of integer dimension $n$, which (in its turn) is defined by a Lagrange fundamental function $L(x,y)$, i.e. a regular real function $L : TM \to \mathbb{R}$, for which the Hessian ${}_Lg_{ij} = (1/2)\partial^2 L/\partial y^i \partial y^j$ is not generated.

Any $\overset{\alpha}{L}(x, {}^\alpha y)$ determines three fundamental geometric objects on $\overset{\alpha}{\underline{L^n}}$:

1. A canonical N-connection ${}_L\overset{\alpha}{\mathbf{N}} = \{{}^\alpha_L N^a_i\}$ structure (with local coefficients[3] ${}^\alpha_L N^a_i$ parametrized for a decomposition ${}_L\overset{\alpha}{\mathbf{N}} = {}^\alpha_L N^a_i(u)(dx^i)^\alpha \otimes \overset{\alpha}{\underline{\partial}}_a$ with respect to local bases (2)

[3]Computed with the aim to encode the fractional Euler–Lagrange equations into a canonical semi-spray configuration [1],

$$
{}^\alpha_L N^a_j = {}_{1y^j}\overset{\alpha}{\underline{\partial}}_j \overset{\alpha}{G}{}^k(x, {}^\alpha y) \quad \text{for } \overset{\alpha}{G}{}^k = \frac{1}{4}{}_L\overset{\alpha}{g}{}^{kj}\left[ y^j_{1y^j}\overset{\alpha}{\underline{\partial}}_j \left( {}_{1x^i}\overset{\alpha}{\underline{\partial}}_i \overset{\alpha}{L} \right) - {}_{1x^i}\overset{\alpha}{\underline{\partial}}_i \overset{\alpha}{L} \right].
$$

and (3)) with an associated class of N-adapted fractional (co) frames linearly depending on ${}^{\alpha}N_i^a$,

$$
{}_{L}^{\alpha}\mathbf{e}_{\beta} = \left[ {}_{L}^{\alpha}\mathbf{e}_j = \overset{\alpha}{\underline{\partial}}_j - {}_{L}^{\alpha}N_j^a \overset{\alpha}{\underline{\partial}}_a, {}^{\alpha}e_b = \overset{\alpha}{\underline{\partial}}_b \right], \tag{4}
$$

$$
{}_{L}^{\alpha}\mathbf{e}^{\beta} = [{}^{\alpha}e^j = (dx^j)^{\alpha}, \qquad {}_{L}^{\alpha}\mathbf{e}^b = (dy^b)^{\alpha} + {}_{L}^{\alpha}N_k^b (dx^k)^{\alpha}]. \tag{5}
$$

2. A canonical (Sasaki type) metric structure,

$$
\begin{aligned}
{}_{L}\overset{\alpha}{\mathbf{g}} &= {}_{L}^{\alpha}g_{kj}(x,y)^{\alpha}e^k \otimes^{\alpha} e^j + {}_{L}^{\alpha}g_{cb}(x,y){}_{L}^{\alpha}\mathbf{e}^c \otimes {}_{L}^{\alpha}\mathbf{e}^b, \\
{}_{L}\overset{\alpha}{g}_{ij} &= \frac{1}{4}\left( \overset{\alpha}{\underline{\partial}}_i \overset{\alpha}{\underline{\partial}}_j + \overset{\alpha}{\underline{\partial}}_j \overset{\alpha}{\underline{\partial}}_i \right) \overset{\alpha}{L} \neq 0,
\end{aligned} \tag{6}
$$

with ${}_{L}^{\alpha}g_{cb}$ computed respectively by the same formulas as ${}_{L}^{\alpha}g_{kj}$.

3. A canonical fractional metrical d-connection ${}_{c}^{\alpha}\mathbf{D} = (h_c^{\alpha}D, v_c^{\alpha}D) = \{{}^{\alpha}\widehat{\boldsymbol{\Gamma}}_{\alpha\beta}^{\gamma} = ({}^{\alpha}\widehat{L}^i_{jk}, {}^{\alpha}\widehat{C}^i_{jc})\}$, where

$$
\begin{aligned}
{}^{\alpha}\widehat{L}^i_{jk} &= \frac{1}{2}{}_{L}^{\alpha}g^{ir}\left( {}_{L}^{\alpha}\mathbf{e}_k {}_{L}^{\alpha}g_{jr} + {}_{L}^{\alpha}\mathbf{e}_j {}_{L}^{\alpha}g_{kr} - {}_{L}^{\alpha}\mathbf{e}_r {}_{L}^{\alpha}g_{jk} \right), \\
{}^{\alpha}\widehat{C}^a_{bc} &= \frac{1}{2}{}_{L}^{\alpha}g^{ad}\left( {}^{\alpha}e_c {}_{L}^{\alpha}g_{bd} + {}^{\alpha}e_c {}_{L}^{\alpha}g_{cd} - {}^{\alpha}e_d {}_{L}^{\alpha}g_{bc} \right)
\end{aligned} \tag{7}
$$

for ${}_{L}^{\alpha}g^{ad}$ being inverse to ${}_{L}^{\alpha}g_{kj}$.

We conclude that the regular fractional mechanics defined by a fractional Lagrangian $\overset{\alpha}{L}$ can be equivalently encoded into canonical geometric data $({}_{L}\overset{\alpha}{\mathbf{N}}, {}_{L}\overset{\alpha}{\mathbf{g}}, {}_{c}^{\alpha}\mathbf{D})$. This allows us to apply a number of powerful geometric methods in fractional calculus and applications.

### 2.3 An Almost Kähler–Lagrange Model of Fractional Mechanics

A fractional nonholonomic almost complex structure can be defined as a linear operator $\overset{\alpha}{\mathbf{J}}$ acting on the vectors on $\overset{\alpha}{\underline{T}}M$ following formulas

$$
\overset{\alpha}{\mathbf{J}}({}_{L}^{\alpha}\mathbf{e}_i) = -{}^{\alpha}e_i \quad \text{and} \quad \overset{\alpha}{\mathbf{J}}({}^{\alpha}e_i) = {}_{L}^{\alpha}\mathbf{e}_i,
$$

where the superposition $\overset{\alpha}{\mathbf{J}} \circ \overset{\alpha}{\mathbf{J}} = -\mathbf{I}$, for $\mathbf{I}$ being the unity matrix. This structure is determined by and adapted to N-connection ${}_{L}\overset{\alpha}{\mathbf{N}}$ induced, in its turn, by a regular fractional $\overset{\alpha}{L}$.

A fractional Lagrangian $\overset{\alpha}{L}$ induces a canonical 1-form

$$
{}_{L}^{\alpha}\omega = \frac{1}{2}\left( {}_{1}y^i \overset{\alpha}{\underline{\partial}}_i \overset{\alpha}{L} \right)^{\alpha} e^i.
$$

Following formula ${}_{L}^{\alpha}\theta(\mathbf{X},\mathbf{Y}) \doteqdot {}_{L}\overset{\alpha}{\mathbf{g}}(\overset{\alpha}{\mathbf{J}}\mathbf{X},\mathbf{Y})$, for any vectors $\mathbf{X}$ and $\mathbf{Y}$ on $\overset{\alpha}{\underline{T}}M$, any metric ${}_{L}\overset{\alpha}{\mathbf{g}}$ (6) determines a canonical 2-form

$$
{}_{L}^{\alpha}\theta = {}_{L}\overset{\alpha}{g}_{ij}(x, {}^{\alpha}y){}_{L}^{\alpha}\mathbf{e}^i \wedge {}^{\alpha}e^j. \tag{8}
$$

The Main Result in [1] (see similar "integer" details in [19, 23, 24]) states that the fractional canonical metrical d-connection ${}_{c}^{\alpha}\mathbf{D}$ with N-adapted coefficients (7), defines a (unique) canonical fractional almost Kähler d-connection ${}^{\theta}_{c}\overset{\alpha}{\mathbf{D}} = {}_{c}^{\alpha}\mathbf{D}$ satisfying the conditions ${}^{\theta}\overset{\alpha}{\mathbf{D}}_{\mathbf{X}L}\overset{\alpha}{\mathbf{g}} = \mathbf{0}$ and ${}^{\theta}\overset{\alpha}{\mathbf{D}}_{\mathbf{X}}\overset{\alpha}{\mathbf{J}} = 0$, for any vector $\mathbf{X} = X^{i\,\alpha}{}_{L}\mathbf{e}_i + X^{a\alpha}e_a$.

The Nijenhuis tensor $\overset{\alpha}{\Omega}$ for $\overset{\alpha}{\mathbf{J}}$ is defined in the form

$$\overset{\alpha}{\mathbf{\Omega}}(\mathbf{X},\mathbf{Y}) \doteqdot \left[\overset{\alpha}{\mathbf{J}}\mathbf{X}, \overset{\alpha}{\mathbf{J}}\mathbf{Y}\right] - \overset{\alpha}{\mathbf{J}}\left[\overset{\alpha}{\mathbf{J}}\mathbf{X}, \mathbf{Y}\right] - \overset{\alpha}{\mathbf{J}}\left[\mathbf{X},\mathbf{Y}\right] - \left[\mathbf{X},\mathbf{Y}\right].$$

A component calculus with respect to N-adapted bases (4) and (5), for $\overset{\alpha}{\mathbf{\Omega}}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = \overset{\alpha\,\gamma}{\mathbf{\Omega}}_{\alpha\beta}\mathbf{e}_\gamma$ results in $\overset{\alpha\,\gamma}{\mathbf{\Omega}}_{\alpha\beta} = 4\overset{\alpha\,\gamma}{\mathbf{T}}_{\alpha\beta}$, where $\overset{\alpha\,\gamma}{\mathbf{T}}_{\alpha\beta}$ is the torsion of an affine fractional connection $\overset{\alpha\,\gamma}{\mathbf{\Gamma}}_{\alpha\beta}$.[4] For ${}_{c}^{\alpha}\mathbf{D} = \{{}^{\alpha}\widehat{\mathbf{\Gamma}}^{\gamma}_{\alpha\beta}\}$, the components of torsion ${}_{L}^{\alpha}\widehat{\mathbf{T}}^{\gamma}_{\alpha\beta}$ are ${}_{L}^{\alpha}\widehat{T}^{i}_{jk} = 0, {}_{L}^{\alpha}\widehat{T}^{a}_{bc} = 0, {}_{L}^{\alpha}\widehat{T}^{i}_{jk} = {}_{L}^{\alpha}\widehat{C}^{i}_{jc}, {}_{L}^{\alpha}\widehat{T}^{a}_{ij} = {}_{L}^{\alpha}\Omega^{a}_{ij}, {}_{L}^{\alpha}\widehat{T}^{a}_{ib} = {}^{\alpha}e_{b}{}_{L}^{\alpha}N^{a}_{i} - {}_{L}^{\alpha}\widehat{L}^{a}_{bi}$.

So, we constructed a canonical (i.e. uniquely determined by $\overset{\alpha}{L}$) almost Kähler distinguished connection (d-connection) ${}^{\theta}\overset{\alpha}{\mathbf{D}}$ being compatible both with the almost Kähler, $({}_{L}^{\alpha}\theta, \overset{\alpha}{\mathbf{J}})$, and N-connection, ${}_{L}^{\alpha}\mathbf{N}$, structures. We can work equivalently with the data $\underline{\overset{\alpha}{L}{}^{n}} = (\underline{\overset{\alpha}{M}}, \overset{\alpha}{L}) = ({}_{L}\overset{\alpha}{\mathbf{N}}, {}_{L}\overset{\alpha}{\mathbf{g}}, {}_{c}^{\alpha}\mathbf{D})$ and/or $\underline{\overset{\alpha}{K}{}^{2n}} = (\overset{\alpha}{\mathbf{J}}, {}_{L}^{\alpha}\theta, {}_{c}^{\alpha}\mathbf{D})$. The last (nonholonomic almost symplectic) ones are most convenient for deformation quantization.

## 3 Fractional Deformations and Quantization

In this section we provide a nonholonomic fractional modification of Fedosov's construction which will be applied for deformation quantization of fractional Lagrange mechanics, see next section.

### 3.1 Star Products for Fractional Symplectic Models

For integer dimensions, any ${}_{L}^{\alpha}\theta$ (8) induces a structure of Poisson brackets $\{\cdot,\cdot\}$ via the Hamilton–Jacobi equations associated to a regular Lagrangian $L$, see details in Corollary 2.1 from Ref. [25]. Working with local fractional Caputo (co) bases (2) and (3), the Poisson structure and derived geometric constructions with data $\underline{\overset{\alpha}{K}{}^{2n}}$ are very similar to those for an abstract, non-singular, Poisson manifold $(V, \{\cdot,\cdot\})$. We shall use the symbol $V$ for a general space (it can be holonomic, or nonholonomic, fractional and/or integer etc.) in order outline some important concepts which, for our purposes, will be latter developed for more rich geometric structures on $V = \underline{\overset{\alpha}{K}{}^{2n}}$.

Let us denote by $C^\infty(V)[[v]]$ the spaces of formal series in variable $v$ with coefficients from $C^\infty(V)$ on a Poisson manifold $(V, \{\cdot,\cdot\})$. A deformation quantization is an associative algebra structure on $C^\infty(V)[[v]]$ with a $v$-linear and $v$-adically continuous star product

$$^{1}f * {}^{2}f = \sum_{r=0}^{\infty} {}_{r}C({}^{1}f, {}^{2}f)v^{r}, \tag{9}$$

[4] This formula is a nonholonomic analog, for our conventions, with inverse sign, of the formula (2.9) from [18].

where ${}_rC, r \geq 0$, are bilinear operators on $C^\infty(V)$ with ${}_0C({}^1f, {}^2f) = {}^1f\,{}^2f$ and ${}_1C({}^1f, {}^2f) - {}_1C({}^2f, {}^1f) = i\{{}^1f, {}^2f\}$, with $i$ being the complex unity.

If all operators ${}_rC, r \geq 0$ are bidifferential, a corresponding star product $*$ is called differential. We can define different star products on a $(V, \{\cdot,\cdot\})$. Two differential star products $*$ and $*'$ are equivalent if there is an isomorphism of algebras $A : (C^\infty(V)[[v]], *) \to (C^\infty(V)[[v]], *')$, where $A = \sum_{r\geq 1}^{\infty} {}_rA v^r$, for ${}_0A$ being the identity operator and ${}_rA$ being differential operators on $C^\infty(V)$.

### 3.2 Fedosov Operators for Fractional Lagrange Spaces

On $\overset{\alpha}{\underline{K}^{2n}}$, we introduce the tensor ${}^{\alpha}_{L}\mathbf{\Lambda}^{\beta\gamma} \doteqdot {}^{\alpha}_{L}\theta^{\beta\gamma} - i\,{}^{\alpha}_{L}\mathbf{g}^{\beta\gamma}$, where $i$ is the complex unity. The local coordinates on $\overset{\alpha}{\underline{T}}M$ are parametrized in the form $u = \{u^\alpha\}$ and the local coordinates on $\underline{T}_u\overset{\alpha}{\underline{T}}M$ are labelled $(u, z) = (u^\gamma, z^\beta)$, where $z^\beta$ are the second order fiber coordinates (we should state additionally a left label $\alpha$ if the fractional character of some coordinates has to be emphasized, for instance to write ${}^\alpha z^\beta$ instead of $z^\beta$). In deformation quantization, there are used formal series

$$a(v, z) = \sum_{r\geq 0, |\overbrace{\beta}| \geq 0} a_{r, \underbrace{\beta}}(u) z^{\overbrace{\beta}} v^r, \tag{10}$$

where $\overbrace{\beta}$ is a multi-index, defining the formal Wick algebra $\overset{\alpha}{\mathbf{W}}_u$. The formal Wick product $\overset{\alpha}{\circ}$ of two elements $a$ and $b$ defined by some formal series (10) is

$$a \overset{\alpha}{\circ} b(z) \doteqdot \exp\left(i\frac{v}{2}\,{}^{\alpha}_{L}\mathbf{\Lambda}^{\alpha\beta}\frac{\partial^2}{\partial z^\alpha \partial z^\alpha_{[1]}}\right) a(z) b(z_{[1]}) \mid_{z=z_{[1]}}. \tag{11}$$

Such a product is determined by a regular fractional Lagrangian $\overset{\alpha}{L}$ and corresponding $\overset{\alpha}{\underline{K}^{2n}}$.

Following the constructions from Refs. [19, 20] for such "d-algebras", we construct a nonholonomic bundle $\overset{\alpha}{\mathbf{W}} = \cup_u\ \overset{\alpha}{\mathbf{W}}_u$ of formal Wick algebras defined as a union of $\mathbf{W}_u$. The fibre product (11) is trivially extended to the space of $\overset{\alpha}{\mathbf{W}}$-valued N-adapted differential forms ${}^{\alpha}_{L}\mathcal{W} \otimes \overset{\alpha}{\Lambda}$ by means of the usual exterior product of the scalar forms ${}^{\alpha}_{L}\mathbf{\Lambda}$, where ${}^{\alpha}_{L}\mathcal{W}$ denotes the sheaf of smooth sections of $\overset{\alpha}{\mathbf{W}}$. There is a standard grading on ${}^{\alpha}_{L}\mathbf{\Lambda}$, denoted $\deg_a$. It is possible to introduce grading $\deg_v, \deg_s, \deg_a$ on ${}^{\alpha}_{L}\mathcal{W} \otimes \overset{\alpha}{\Lambda}$ defined on homogeneous elements $v, z^\beta, {}^{\alpha}_{L}\mathbf{e}^\beta$ as follows: $\deg_v(v) = 1$, $\deg_s(z^\alpha) = 1$, $\deg_a({}^{\alpha}_{L}\mathbf{e}^\alpha) = 1$, and all other gradings of the elements $v, z^\alpha, {}^{\alpha}_{L}\mathbf{e}^\alpha$ are set to zero (we adapt to nonholonomic fractional configuration the conventions from [18–20]). We extend the canonical d-connection ${}^{\alpha}_{c}\mathbf{D} = \{{}^\alpha\widehat{\mathbf{\Gamma}}^\gamma_{\alpha\beta}\}$ (7) to an operator on ${}^{\alpha}_{L}\mathcal{W} \otimes \overset{\alpha}{\Lambda}$ following the formula

$${}^\alpha\check{\mathbf{D}}\,(a \otimes \lambda) \doteqdot \left({}^{\alpha}_{L}\mathbf{e}_\alpha(a) - u^\beta\,{}^\alpha\widehat{\mathbf{\Gamma}}^\gamma_{\alpha\beta}\,{}^z\mathbf{e}_\alpha(a)\right) \otimes ({}^{\alpha}_{L}\mathbf{e}^\alpha \wedge \lambda) + a \otimes d\lambda,$$

where ${}^z\mathbf{e}_\alpha$ is a similar to ${}^{\alpha}_{L}\mathbf{e}_\alpha$ on N-anholonomic fibers of $\underline{T}\overset{\alpha}{\underline{T}}M$, depending on $z$-variables (for holonomic second order fibers, we can take ${}^z\mathbf{e}_\alpha = \partial/\partial z^\alpha$).

**Definition 3.1** The Fedosov distinguished operators (d-operators) ${}_{L}^{\alpha}\delta$ and ${}_{L}^{\alpha}\delta^{-1}$ on ${}_{L}^{\alpha}\mathcal{W}\otimes\overset{\alpha}{\Lambda}$ are

$$
\begin{aligned}
{}_{L}^{\alpha}\delta(a) &= {}_{L}^{\alpha}\mathbf{e}^{\alpha}\wedge{}^{z}\mathbf{e}_{\alpha}(a),\\
{}_{L}^{\alpha}\delta^{-1}(a) &= \begin{cases} \frac{i}{p+q}z^{\alpha\alpha}{}_{L}^{\alpha}\mathbf{e}_{\alpha}(a), & \text{if } p+q>0,\\ 0, & \text{if } p=q=0,\end{cases}
\end{aligned}
\tag{12}
$$

where $a\in{}_{L}^{\alpha}\mathcal{W}\otimes\overset{\alpha}{\Lambda}$ is homogeneous w.r.t. the grading $\deg_s$ and $\deg_a$ with $\deg_s(a)=p$ and $\deg_a(a)=q$.

The d-operators (12) satisfy the property that

$$
a=({}_{L}^{\alpha}\delta{}_{L}^{\alpha}\delta^{-1}+{}_{L}^{\alpha}\delta^{-1}{}_{L}^{\alpha}\delta+\sigma)(a),
$$

where $a\longmapsto\sigma(a)$ is the projection on the $(\deg_s,\deg_a)$-bihomogeneous part of $a$ of degree zero, $\deg_s(a)=\deg_a(a)=0$. We can verify that ${}_{L}^{\alpha}\delta$ is also a $\deg_a$-graded derivation of d-algebra $({}_{L}^{\alpha}\mathcal{W}\otimes\overset{\alpha}{\Lambda},\overset{\alpha}{\circ})$.

A fractional Lagrangian $\overset{\alpha}{L}$ induces respective torsion and curvature

$$
\begin{aligned}
\widehat{\mathcal{T}} &\doteqdot \frac{z^{\gamma}}{2}{}_{L}^{\alpha}\theta_{\gamma\tau}{}_{L}^{\alpha}\widehat{\mathbf{T}}^{\tau}_{\alpha\beta}(u){}_{L}^{\alpha}\mathbf{e}^{\alpha}\wedge{}_{L}^{\alpha}\mathbf{e}^{\beta},\\
\widehat{\mathcal{R}} &\doteqdot \frac{z^{\gamma}z^{\varphi}}{4}{}_{L}^{\alpha}\theta_{\gamma\tau}{}_{L}^{\alpha}\widehat{\mathbf{R}}^{\tau}_{\varphi\alpha\beta}(u){}_{L}^{\alpha}\mathbf{e}^{\alpha}\wedge{}_{L}^{\alpha}\mathbf{e}^{\beta},
\end{aligned}
$$

on ${}_{L}^{\alpha}\mathcal{W}\otimes\overset{\alpha}{\Lambda}$, for ${}_{L}^{\alpha}\widehat{\mathbf{T}}^{\gamma}_{\alpha\beta}$ and ${}_{L}^{\alpha}\widehat{\mathbf{R}}^{\tau}_{\varphi\alpha\beta}$ being respectively the torsion and curvature of the canonical d-connection ${}_{c}^{\alpha}\mathbf{D}$ (7).

Using the formulas (10) and (11) and the identity

$$
{}_{L}^{\alpha}\theta_{\varphi\tau}{}_{L}^{\alpha}\widehat{\mathbf{R}}^{\tau}_{\gamma\alpha\beta}={}_{L}^{\alpha}\theta_{\gamma\tau}{}_{L}^{\alpha}\widehat{\mathbf{R}}^{\tau}_{\varphi\alpha\beta},
\tag{13}
$$

we prove the important formulas:

**Proposition 3.1** *The fractional Fedosov d-operators satisfy the properties*

$$
\left[{}^{\alpha}\check{\mathbf{D}},{}_{L}^{\alpha}\delta\right]=\frac{i}{v}ad_{\mathrm{Wick}}(\widehat{\mathcal{T}})\quad\text{and}\quad{}^{\alpha}\check{\mathbf{D}}^{2}=-\frac{i}{v}ad_{\mathrm{Wick}}(\widehat{\mathcal{R}}),
\tag{14}
$$

*where* $[\cdot,\cdot]$ *is the* $\deg_a$*-graded commutator of endomorphisms of* ${}_{L}^{\alpha}\mathcal{W}\otimes\overset{\alpha}{\Lambda}$ *and* $ad_{\mathrm{Wick}}$ *is defined via the* $\deg_a$*-graded commutator in* $({}_{L}^{\alpha}\mathcal{W}\otimes\overset{\alpha}{\Lambda},\overset{\alpha}{\circ})$.

We have all formal geometric ingredients for performing deformation quantization of fractional Lagrange mechanics.

## 4 Fedosov Quantization of Fractional Mechanics

The Fedosov's deformation quantization theory is generalized to fractional Lagrange spaces. The class $c_0$ of the deformation quantization of fractional Lagrange geometry is calculated.

### 4.1 Main Theorems for Fractional Lagrange Spaces

We denote the *Deg*-homogeneous component of degree $k$ of an element $a \in {}_L^\alpha\mathcal{W} \otimes \overset{\alpha}{\Lambda}$ by $a^{(k)}$.

**Theorem 4.1** *For any regular fractional Lagrangian $\overset{\alpha}{\mathrm{L}}$ and corresponding canonical almost Kähler–Lagrange model $\underline{\overset{\alpha}{K}^{2n}} = (\overset{\alpha}{\mathbf{J}}, {}_L^\alpha\theta, {}_c^\alpha\mathbf{D})$, there is a flat canonical fractional Fedosov d-connection*

$$ {}_L^\alpha\widehat{\mathcal{D}} \doteqdot -{}_L^\alpha\delta + {}^\alpha\check{\mathbf{D}} - \frac{i}{v} ad_{Wick}(r) $$

*satisfying the condition ${}_L^\alpha\widehat{\mathcal{D}}^2 = 0$, where the unique element $r \in {}_L^\alpha\mathcal{W} \otimes \overset{\alpha}{\Lambda}$, $\deg_a(r) = 1$, ${}^\alpha\check{\mathbf{D}}_L^\alpha \delta^{-1} r = 0$, solves the equation*

$$ {}_L^\alpha\delta r = \widehat{\mathcal{T}} + \widehat{\mathcal{R}} + {}^\alpha\check{\mathbf{D}} r - \frac{i}{v} r \overset{\alpha}{\circ} r $$

*and this element can be computed recursively with respect to the total degree Deg as follows*:

$$ \begin{aligned} r^{(0)} &= r^{(1)} = 0, \qquad r^{(2)} = {}_L^\alpha\delta^{-1}\widehat{\mathcal{T}}, \\ r^{(3)} &= {}_L^\alpha\delta^{-1}\left(\widehat{\mathcal{R}} + {}^\alpha\check{\mathbf{D}} r^{(2)} - \frac{i}{v} r^{(2)} \overset{\alpha}{\circ} r^{(2)}\right), \\ r^{(k+3)} &= {}_L^\alpha\delta^{-1}\left({}^\alpha\check{\mathbf{D}} r^{(k+2)} - \frac{i}{v}\sum_{l=0}^{k} r^{(l+2)} \overset{\alpha}{\circ} r^{(l+2)}\right), \quad k \geq 1. \end{aligned} $$

*Proof* The proof is similar to the standard Fedosov constructions if we work with the Caputo fractional derivative in N-adapted form, by induction using the identities

$$ {}_L^\alpha\delta\widehat{\mathcal{T}} = 0 \quad \text{and} \quad {}_L^\alpha\delta\widehat{\mathcal{R}} = {}^\alpha\check{\mathbf{D}}\widehat{\mathcal{T}}. $$

For integer dimensions and holonomic configurations we get the results from Ref. [18] proved for arbitrary affine connections with torsion and almost Kähler structures on $M$. □

The next theorem gives a rule how to define and compute the star product (which is the main purpose of deformation quantization) induced by a regular fractional Lagrangian.

**Theorem 4.2** *A star-product $\overset{\alpha}{*}$ on the canonical almost Kähler model of fractional Lagrange space $\underline{\overset{\alpha}{K}^{2n}} = (\overset{\alpha}{\mathbf{J}}, {}_L^\alpha\theta, {}_c^\alpha\mathbf{D})$ is defined on $C^\infty(\underline{\overset{\alpha}{L}^n})[[v]]$ by formula*

$$ {}^1f \overset{\alpha}{*} {}^2f \doteqdot \sigma(\tau({}^1f)) \overset{\alpha}{\circ} \sigma(\tau({}^2f)), $$

*where the projection $\sigma : {}_L^\alpha\mathcal{W}_{\widehat{\mathcal{D}}} \to C^\infty(\underline{\overset{\alpha}{L}^n})[[v]]$ onto the part of* $\deg_s$*-degree zero is a bijection and the inverse map $\tau : C^\infty(\underline{\overset{\alpha}{L}^n})[[v]] \to {}_L^\alpha\mathcal{W}_{\widehat{\mathcal{D}}}$ is calculated recursively w.r.t. the total degree Deg*,

$$ \tau(f)^{(0)} = f \quad \textit{and}, \quad \textit{for } k \geq 0, $$

$$\tau(f)^{(k+1)} = {}^{\alpha}_{L}\delta^{-1}\left({}^{\alpha}\check{\mathbf{D}}\tau(f)^{(k)} - \frac{i}{v}\sum_{l=0}^{k} ad_{Wick}(r^{(l+2)})(\tau(f)^{(k-l)})\right).$$

*Proof* We may check by explicit computations similarly to those in [18], in our case, with fractional Caputo derivatives that such constructions give a well defined star product. □

### 4.2 Cohomology Classes of Quantized Fractional Lagrangians

The characteristic class of a star product is $(1/iv)[\theta] - (1/2i)\varepsilon$, where $\varepsilon$ is the canonical class for an underlying Kähler manifold, for nonholonomic Lagrange–Einstein–Finsler spaces we analysed this construction in Refs. [19–22]. This canonical class can be defined for any almost complex manifold. We show how such a calculus of the crucial part of the characteristic class $cl$ of the fractional star product $\overset{\alpha}{*}$ from Theorem 4.2 can be performed. In explicit form, we shall compute the coefficient $c_0$ at the zeroth degree of $v$.

A straightforward computation of ${}_2C$ from (9), using statements of Theorem 4.1, results in a proof of

**Lemma 4.1** *There is a unique fractional* 2*-form* ${}^{\alpha}_{L}\varkappa$ *which can be expressed*

$${}^{\alpha}_{L}\varkappa = -\frac{i}{8}{}^{\alpha}_{L}\mathbf{J}^{\gamma'}_{\tau}\,{}^{\alpha}_{L}\widehat{\mathbf{R}}^{\tau}_{\gamma'\gamma\beta}\,{}^{\alpha}_{L}\mathbf{e}^{\gamma}\wedge{}^{\alpha}_{L}\mathbf{e}^{\beta} - i\,{}^{\alpha}_{L}\lambda,$$

*where the exact* 1*-form* ${}^{\alpha}_{L}\lambda = d\,{}^{\alpha}_{L}\mu$, *for* ${}^{\alpha}_{L}\mu = \frac{1}{6}{}^{\alpha}_{L}\mathbf{J}^{\alpha'}_{\tau}\,{}^{\alpha}_{L}\widehat{\mathbf{T}}^{\tau}_{\alpha'\beta}\,{}^{\alpha}_{L}\mathbf{e}^{\beta}$, *with nontrivial components of curvature and torsion defined by the canonical d-connection.*

This allows us to compute the Chern–Weyl form

$$\begin{aligned}{}^{\alpha}_{L}\gamma &= -i\mathrm{Tr}\left[(h\Pi, v\Pi)^{\alpha}_{L}\widehat{\mathbf{R}}(h\Pi, v\Pi)^{T}\right] = -i\mathrm{Tr}\left[(h\Pi, v\Pi)^{\alpha}_{L}\widehat{\mathbf{R}}\right]\\ &= -\frac{1}{4}{}^{\alpha}_{L}\mathbf{J}^{\alpha'}_{\tau}\,{}^{\alpha}_{L}\widehat{\mathbf{R}}^{\tau}_{\alpha'\alpha\beta}\,{}^{\alpha}_{L}\mathbf{e}^{\alpha}\wedge{}^{\alpha}_{L}\mathbf{e}^{\beta}\end{aligned}$$

to be closed. By definition, the canonical class is ${}^{\alpha}_{L}\varepsilon \doteqdot [{}^{\alpha}_{L}\gamma]$.[5] These formulas and Lemma 4.1 give the proof of

**Theorem 4.3** *The zero-degree cohomology coefficient for the almost Kähler model of fractional Lagrange space* $\underline{\overset{\alpha}{L}^{n}}$ *is computed* $c_0(\overset{\alpha}{*}) = -(1/2i){}^{\alpha}_{L}\varepsilon$, *where the value* ${}^{\alpha}_{L}\varepsilon$ *is canonically defined by a regular fractional Lagrangian* $\overset{\alpha}{L}(u)$.

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[5] For simplicity, we recall the definition of the canonical class $\varepsilon$ of an almost complex manifold $(M, \mathbb{J})$ of integer dimension and redefine it for ${}^{N}TTM = hTM \oplus vTM$. The distinguished complexification of such second order tangent bundles is introduced in the form $T_{\mathbb{C}}({}^{N}TTM) = T_{\mathbb{C}}(hTM) \oplus T_{\mathbb{C}}(vTM)$. For such nonholonomic bundles, the class ${}^{N}\varepsilon$ is the first Chern class of the distributions $T'_{\mathbb{C}}({}^{N}TTM) = T'_{\mathbb{C}}(hTM) \oplus T'_{\mathbb{C}}(vTM)$ of couples of vectors of type $(1,0)$ both for the h- and v-parts. We can calculate both for integer and fractional dimensions the canonical class ${}^{L}\varepsilon$ (we put the label $L$ for the constructions canonically defined by a regular Lagrangian $L$) for the almost Kähler model of a Lagrange space $L^n$. We take the canonical d-connection ${}^{L}\widehat{\mathbf{D}}$ that it was used for constructing $*$ and considers h- and v-projections $h\Pi = \frac{1}{2}(Id_h - iJ_h)$ and $v\Pi = \frac{1}{2}(Id_v - iJ_v)$, where $Id_h$ and $Id_v$ are respective identity operators and $J_h$ and $J_v$ are projection operators onto corresponding $(1,0)$-subspaces. The matrix $(h\Pi, v\Pi)\widehat{\mathbf{R}}(h\Pi, v\Pi)^T$, where $(\ldots)^T$ denotes the transposition, is the curvature matrix of the restriction of the connection ${}^{L}\widehat{\mathbf{D}}$ to $T'_{\mathbb{C}}({}^{N}TTM)$. For fractional dimensions, such formulas "obtain" corresponding let labels with $\alpha$.

Finally we note that the formula from this Theorem can be directly applied for the Cartan connection in Finsler geometry with $\overset{\alpha}{L} = (\overset{\alpha}{F})^2$, where $\overset{\alpha}{F}$ is the fundamental generating Finsler function in fractional Finsler geometry, and in certain fractional generalizations of Einstein and Ricci flow theories [2, 3].

## 5 Conclusions and Discussion

In this paper we provided a generalization of Fedosov's method for quantizing the fractional Lagrange mechanics with Caputo fractional derivatives. We used a fundamental result that nonholonomic geometries (for certain classes of integro-differential distributions modeling fractional spaces [2, 3]) can be modeled as some almost Kähler configurations which can be quantized following Karabegov and Schlichenmaier ideas [18].

We argue that the approach to fractional calculus based on Caputo fractional derivative is a self-consistent comprehensive one allowing geometrization of fundamental field and evolution equations and their quantization at least in the meaning of deformation quantization theory.

In various directions of modern mathematics, physics, mathematical economics etc, there are also considered, and preferred, different fractional derivatives, for instance, the Riemann–Liouville (RL) derivative. It is a problem, at least technically, to elaborate a well defined differential geometry with RL type fractional derivatives not resulting in zero acting on constants (see detailed discussions in [2, 3]). So, for such fractional calculus approaches we can not geometrize mechanical and field/evolution interactions in a standard form. In general, it is not clear how to define a RL-differential geometry which would mimic certain integer dimension type geometries. As a result, we can not perform a RL-quantization following usual geometric/deformation methods.

Our proposal, is that for fractional models, for instance, with RL fractional derivative, we can geometrize the constructions, and elaborate quantum models taking the Caputo derivatives for certain background constructions and then to deform nonholonomically the geometric objects in order to re-adapt them and generate a necessary RL, or another ones, fractional theory.

**Acknowledgements** S.V. is grateful to Çankaya University for support of his research on fractional calculus, geometry and applications.

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